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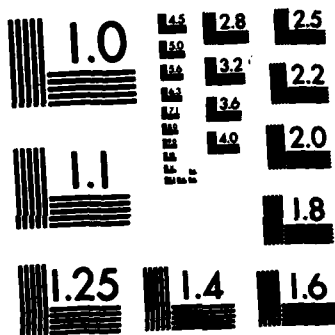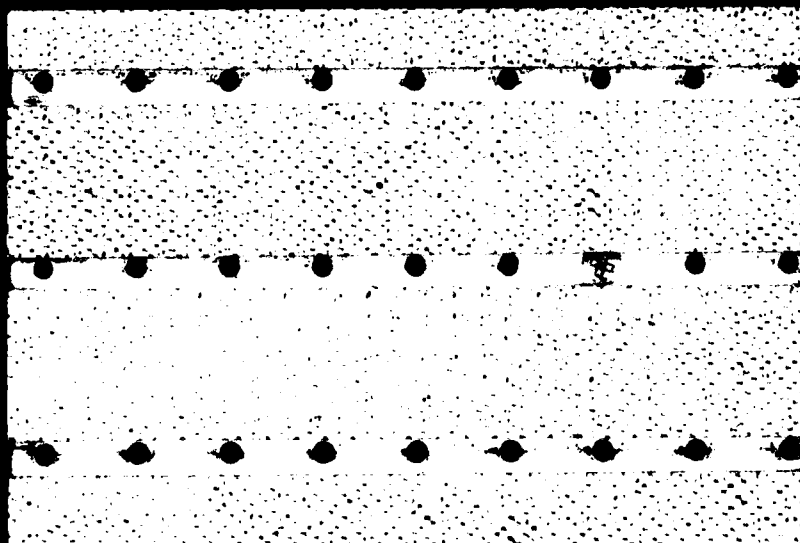DTIC

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OPTIMAL LOT-SIZING IN ACYCLIC

MULTIPERIOD PRODUCTION SYSTEMS

by

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ABSTRACT

This paper presents constructive, network-based proofs for the Wagner-Whitin property and a generalized Nested Schedule property for optimal production schedules in acyclic hierarchical multiperiod production systems. Algorithms for obtaining schedules with these properties and described and illustrated with examples.

INTRODUCTION

This paper addresses the problem of lot-sizing in hierarchical pro-
duction systems. Specifically, we use a constructive, network based method
to establish the Wagner-Whiten (WW) property and a generalization of the
nested schedules (NS) property for optimal production schedules in unca-
pacitated acyclic multistage systems. In establishing these properties,
we use a generalized network flow model similar to that of Steinberg and
Napier [4].

In the problem we address here, there are N production stages requiring
production decisions for each of T periods in the planning horizon. For
any stage i, $P(i)$ in the index set for predecessor stages and $S(i)$ is the
index set for successor stages. If $S(i) = \phi$, then stage i corresponds to
a final product, and if $P(i) = \phi$, then stage i corresponds to the initial
processing of raw material. Define $F = \{i | P(i) = \phi\}$, the set of initial
(or _first_) production stages and $L = \{i | S(i) = \phi\}$, the set of final (or
last) stages. The production of one unit at stage j, $j \notin F$, requires K_{ij}
units from stage i, $i \in P(j)$ where K_{ij} is a positive integer. For stages
$i \in L$, the demand in period t is D_{it}.

The decisions required are denoted by X_{it}, representing the number of
units to produce at stage i in period t. The end of period inventories
are denoted by I_{it}. The costs associated with production and inventory are
assumed to be concave function and are denoted by $C_{it}(X_{it})$ and $H_{it}(I_{it})$
respectively. Note that $C_{it}(X_{it})$ can incorporate a fixed, or setup cost.

We adopt several standard assumptions, namely:

(1) initial inventories and production lead times are zero at
each stage;

(2) transfer times between stages are zero;

(3) backlogging or lost sales are not permitted; and

(4) there are no capacity limits on production or inventory.

Also, for convenience, we assume that if $j \in S(i)$, then $j > i$. Let P denote any problem having the structure just described.

THE NETWORK MODEL

Figure 1 illustrates the logical structure of a five stage acyclic problem and Figure 2 illustrates the corresponding network model for a four period horizon. In the network there are two nodes for each stage-period combination. The first, referred to as a <u>collector node</u>, corresponds to obtaining units from the predecessor stages, while the second, referred to as a <u>distributor node</u>, corresponds to disposition of current production and the inventory from the previous period. <u>Production arcs</u> join collector nodes to distributor nodes, and their flows correspond to X_{it}. <u>Inventory arcs</u> join distributor nodes, and their flows correspond to I_{it}. <u>Transfer arcs</u> join distributor nodes to collector nodes, and their flows correspond to $K_{ji}X_{it}$, $j \in P(i)$.

The nodes in the flow network are numbered in a particular order, which is used later in the proofs. At each stage, all the collector nodes have consecutive indices, and all the distributor nodes have consecutive indices with smaller indices corresponding to earlier time periods. Moreover, a production arc starting at collector node i will terminate at distributor node i + T. The first collector node for stage n will have index $2(n-1)T + 1$ and the last distributor node will have index $2nT$.
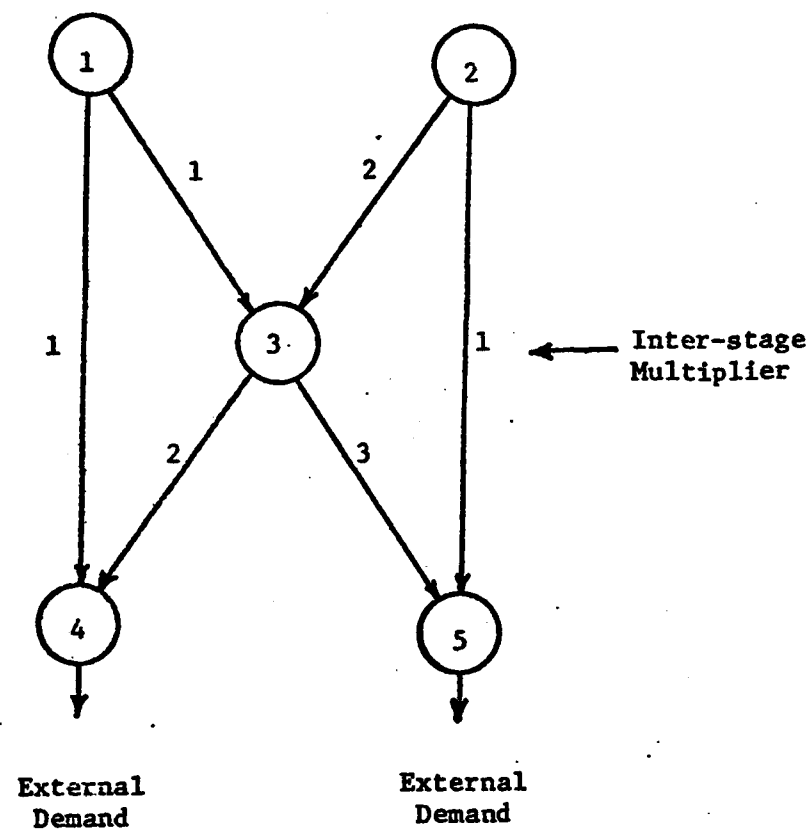
Figure 1. A 4-Stage Acyclic Structure
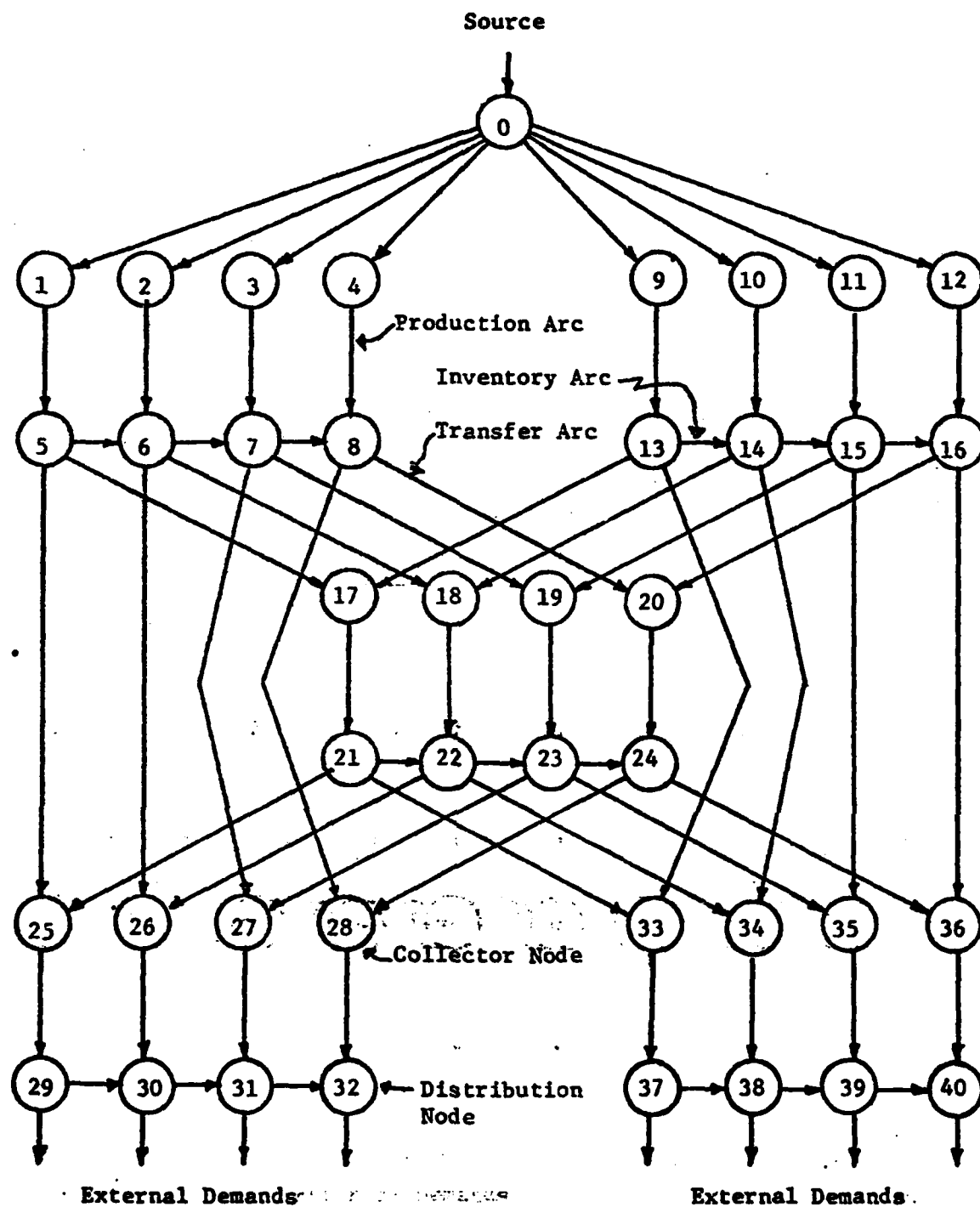
Figure 2. Network model for the 4-stage acyclic system
of Figure 1, with a planning horizon of 4 periods.

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For arc (m, n) in the network, the notation is:

$X(m, n)$ = flow on the arc

$K(m, n)$ = arc multiplier

$C_{(m, n)}(X(m, n)$ = cost of arc flow.

In addition, for node m in the network,

$i(m)$ = the stage corresponding to node m

$t(m)$ = the period corresponding to node m.

A flow in the network is feasible provided:

(1) at each distributor node, total incoming flow is equal to total outgoing flow; and

(2) at each collector node, the incoming flow on an arc is equal to the arc multiplier (K_{ij}) multiplied by the outgoing flow.

There is an obvious correspondence between feasible flows in the network and feasible production schedules. For any problem, P, let $N(P)$ be the corresponding network.

Steinberg and Napier [4] first utilized a network model like this as the basis for a MIP formulation of the problem. Subsequently, McClain et al., [3] demonstrated a more parsimonious mathematical formulation, but did not perform any computational assessment. Our objective is to use the network $N(P)$ to establish two conditions that are satisfied by some optimal solution to P.

THE WAGNER-WHITIN PROPERTY

The Wagner-Whitin property was first established for single-stage problems [7], and later extended to acyclic production systems by Zangwill [8] and Veinott [5]. In essence, it says there is at least one optimal solution for which production and incoming inventory are not both positive in any stage-period. Yet another proof for the property is offered here because: (1) our proof is constructive and network-based, thus more intuitive than the analytic proofs of [5] and [8]; and (2) the flow adjustment operation used in the proof is also required in our proof of a generalized nested schedules property.

Theorem 1. There exists an optimal solution to any problem satisfying

$$X_{it}I_{i,t-1} = 0 \quad \text{for} \quad i = 1,\ldots, N$$
$$\text{and} \quad t = 1,\ldots, T$$

Theorem 1 can be restated in terms of the network $N(P)$ as follows:

Theorem 1N: There exists an optimal flow in $N(P)$ such that at most one arc entering each distributor node has a non-zero flow.

We shall prove the network version of the theorem. Our method of proof is similar to that used by Wagner [7], i.e., we assume that an optimal flow is given which violates the theorem and show how to modify it to obtain a flow satisfying the theorem without increasing the cost.

To start with, we have a given feasible flow that does not satisfy the WW property. We consider the distributor nodes in the order of increasing index, and stop with the first one violating the WW property. For this node, let the incoming flow on the production arc be U units and that on the in-

ventory arc be V units. We consider two alternatives for adjusting the flow: (1) to adjust the flow such that the flow on the production arc vanishes and that on the inventory arc becomes U+V, or (2) to adjust the flow such that the flow on the inventory arc vanishes and that on the production arc becomes U+V.

The flow adjustment is accomplished by creating an adjusting-flow _pattern_ which is itself feasible at every node. The cost associated with the pattern will be concave, thus one of the two alternatives will yield a cost less than or equal to the original cost.

The operation of adjusting the flow at a given node which violates the WW property is called "operation-WWA". In addition to yielding a WW-flow at that node, the operation must ensure that the adjusted flow is feasible, and that the nodes which satisfied the WW property before the operation must continue to satisfy the property after the operation.

Suppose we are adjusting the flow at distributor node J. The adjusting flow, since it must be feasible, should correspond to a "generalized cycle of flow". It will be necessary to identify not only the set of arcs in this generalized cycle, but also the direction of adjusting flow in each arc. We construct, to serve this purpose, two subgraphs, $\Gamma_1(J)$ and $\Gamma_2(J)$, as a part of operation WWA. $\Gamma_1(J)$ contains all the arcs in the network whose flows contribute to the flow on the production arc at J and similarly, $\Gamma_2(J)$ contains the arcs whose flows contribute to the flow on the inventory arc at J. The two subgraphs need not be disjoint.

In a given feasible flow if we want to introduce changes in the inflow at J, corresponding changes have to be introduced on arcs in $\Gamma_1(J)$ and $\Gamma_2(J)$ in order that feasibility of flow is maintained. To satisfy feasibility,

the changes in the production arc and the entering inventory arc at J are always equal in magnitude and opposite in sign. Changes in these two arc flows induce changes in $\Gamma_1(J)$ and $\Gamma_2(J)$ respectively.

The example network of Figure 2 will be used to illustrate these notions. Table 1 presents a set of arc costs, and Figure 3 contains a feasible flow which violates the WW property at node 23. The subgraphs, $\Gamma_1(23)$ and $\Gamma_2(23)$ are shown in Figure 4. It should be evident from this figure that the changes to arc flows can be computed "backward" in $\Gamma_1(23)$ and $\Gamma_2(23)$ starting with the inventory or production arc and using the arc multipliers. Figure 5 shows the adjusted flow.

The algorithm for constructing a WW flow is given in Appendix A. The algorithm makes use of a set of labels, $\ell_i(m, n)$ defined for arcs $(m, n) \in \Gamma_i(J)$. A flow change of Δ on arc $(J-T, T)$ induces a change of $\ell_1(m, n) \cdot \Delta$ for $(m, n) \in \Gamma_1(J)$ and $-\ell_2(m, n) \cdot \Delta$ for $(m, n) \in \Gamma_2(J)$. Note that, as long as the WW property is satisfied at all distributor nodes with indices smaller than J, if an arc (m, n) is in both $\Gamma_1(J)$ and $\Gamma_2(J)$ then $\ell_1(m, n) = \ell_2(m, n)$. Thus, the net flow change for such an arc will be zero. Also, the adjusted flow cost is computed by:

$$F(\Delta) = \sum_{(m, n) \in \Gamma_1(J) \cup \Gamma_2(J)} C_{(m, n)}(X(m, n) + (\ell_1(m, n) - \ell_2(m, n))\Delta)$$

Thus, F(0) is the current cost for arcs in the two subgraphs.

Proof of Theorem 1N

Let J be a distributor node such that the WW property is violated by J, and is satisfied by every distributor node less than J. Then operation WWA, when applied at J adjusts the flow such that:

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TABLE 1

(i,t)	1,1	1,2	1,3	1,4	2,1	2,2	2,3	2,4	3,1	3,2	3,3	3,4	4,1	4,2	4,3	4,4	5,1	5,2	5,3	5,4
F(i,t)	100	80	60	40	100	80	60	40	10	10	10	10	100	100	90	90	90	80	80	80
c(i,t)	10	8	6	4	10	8	6	4	1	1	1	1	10	10	10	8	9	9	9	8
h(i,t)	5	5	2	–	5	5	2	–	16	16	16	–	40	40	40	–	60	60	60	–

Cost data for the acyclic system in Example 2.

The cost structure is: $C(i,t) = F(i,t) + c(i,t)\ X(i,t)$ for $X(i,t) > 0$, and $C(i,t) = 0$ otherwise; $H(i,t) = h(i,t)\ I(i,t)$.
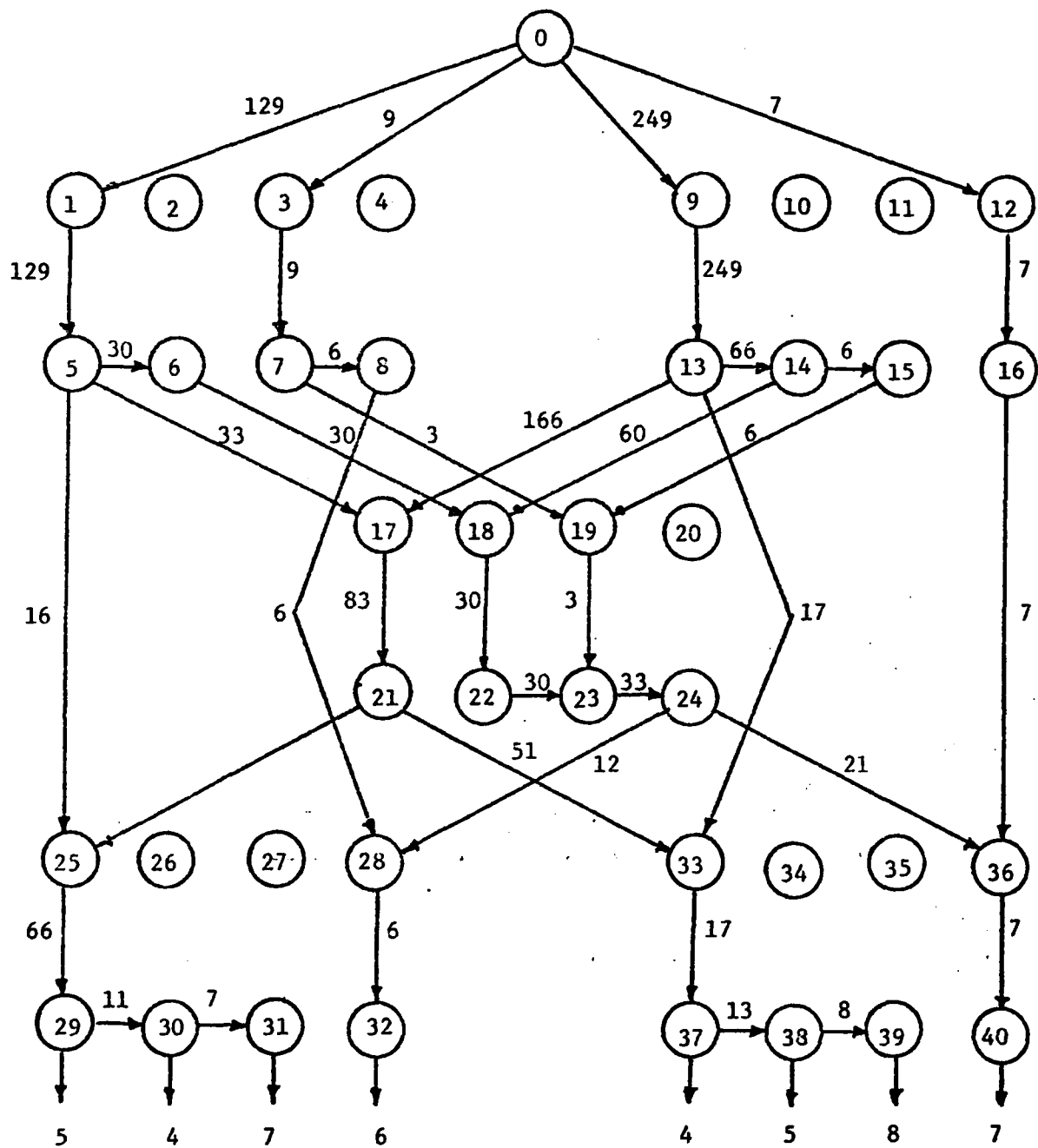
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Figure 3. Feasible Flow Violating WW Property

Legend

o———o $\Gamma_2(23)$
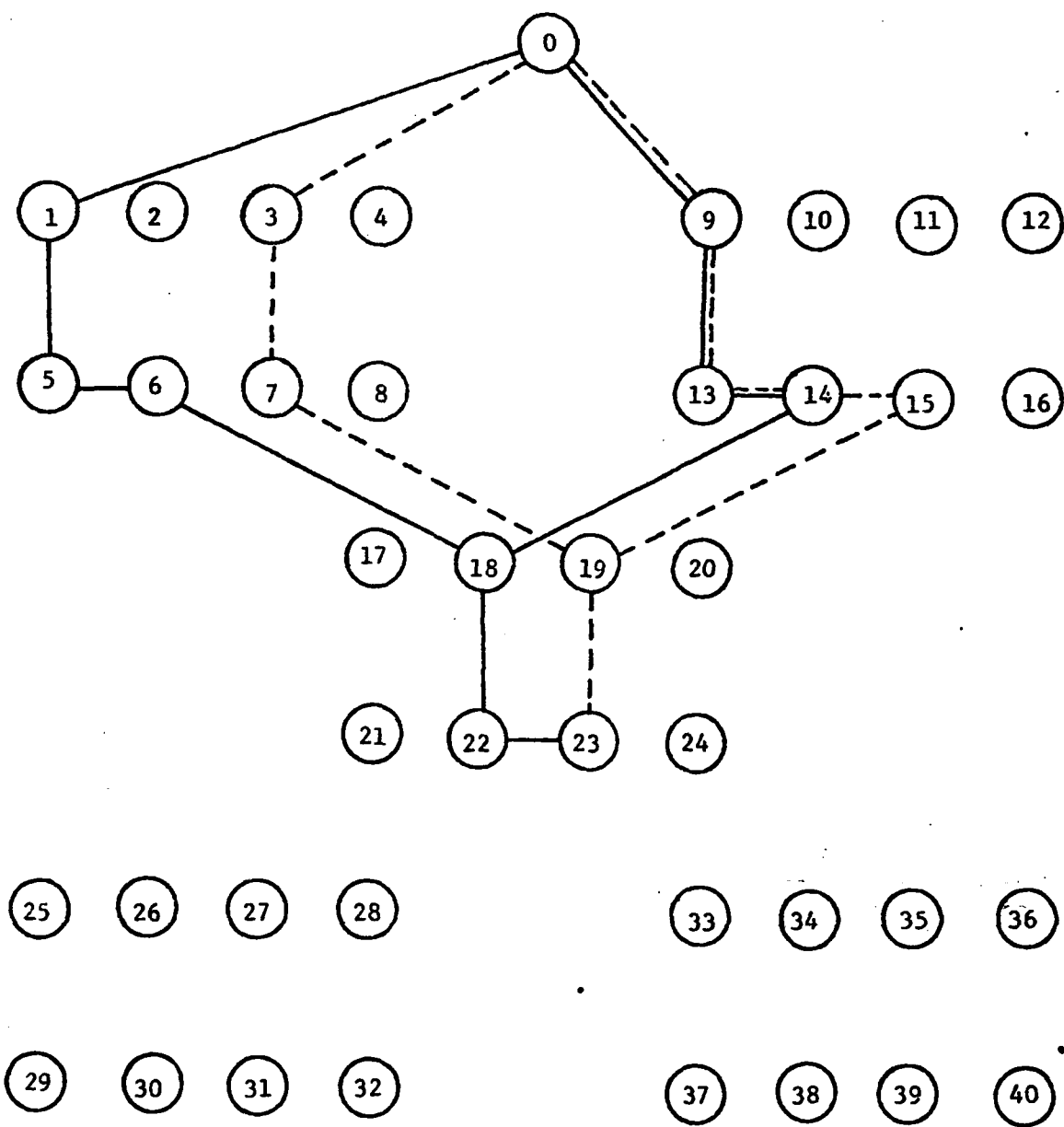
o----o $\Gamma_1(23)$

Figure 4. Subgraphs for Flow of Figure 3

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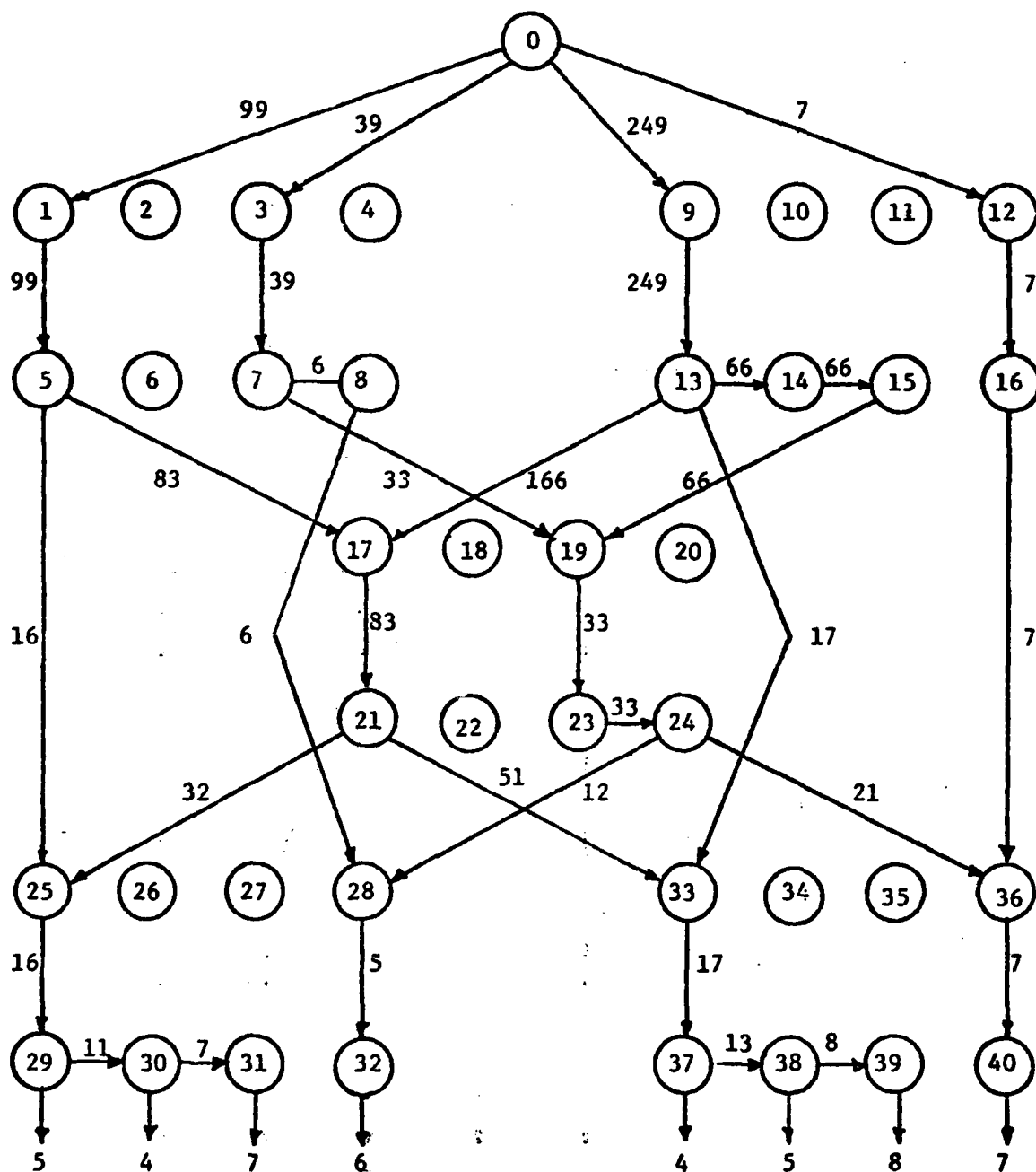
Figure 5. Adjusted Flow

1. In the adjusted flow, the WW property holds at node J and also at all distributor nodes with index less than J.

2. The adjusted flow at nodes greater than J is the same as the flow at those nodes before the application of the operation.

3. The adjusted flow is feasible.

4. The cost of the adjusted flow is less than or equal to the original cost.

The first two of the above four properties are obvious and hence we omit a formal proof for those properties.

The third property follows from the fact that the adjusting-flow in subgraph $\Gamma_1(J) \cup \Gamma_2(J)$ is feasible. Therefore, the adjusted flow which is the sum of the original and the adjusting-flows should be feasible also.

To establish the fourth property, we compute the cost changes in $\Gamma_1(J) \cup \Gamma_2(J)$ and show that the total cost along those arcs either decreases or remains the same as the original cost.

First, note that $C_{(m, n)}(X)$ is a concave function of X, therefore $C_{(m, n)}(k + y)$ is a concave function of y for fixed k. Thus $F(\Delta)$ is a sum of concave functions and therefore also concave. From the properties of concave function, we have:

$$F(1) > F(0) \Rightarrow F(-U) < F(0) \quad \text{and}$$
$$F(1) \leq F(0) \Rightarrow F(V) \leq F(0).$$

It follows that the cost of the adjusted flow will not exceed the cost of the original flow. Q.E.D.

THE NESTED SCHEDULE PROPERTY

A nested schedule property has been studied by Love [2] for pure series systems and by Crowston and Wagner [1] for pure assembly and general acyclic systems. For acyclic systems, a nested schedule is a production schedule for which production at any stage $i \notin L$ and any period t is always accompanied by production in period t at some immediate successor of stage i.

Crowston and Wagner limited their discussion to the case of unit inter-stage multipliers and linear inventory holding costs. We allow general integer valued multipliers and concave inventory costs, and offer a network-based proof.

Theorem 2. An N-stage, T-period problem, P, has at least one optimal solution that simultaneously satisfies the Wagner-Whitin and nested schedules properties provided:

1. The production costs for every stage-period are non-negative, concave, and zero for zero production.

2. Within a stage the production costs are nonincreasing in time. That is

$$C_{i,\ 1}(X) \geqq C_{i,\ 2}(X) \geqq \ldots \geqq C_{i,\ T-1}(X) \geqq C_{i,\ T}(X)$$

for all i

3. The inventory holding cost for a stage-period is non-negative and concave with respect to the end-of-the-period inventory, and zero when the end-of-the-period inventory is zero.

4. The inventory holding costs are non-decreasing in stages, and, additionally, satisfy the following property:

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$$H_{i,\,t}(X) \overset{\geq}{=} \sum_{j\,\epsilon\,P(i)} H_{j,\,t}(K_{ji}\,X)$$

for i ∉ F, all t.

This theorem can also be restated in terms of the network model:

Theorem 2N: Under conditions (1)-(4) of Theorem 2, there is at least one

optimal flow in the network which satisfies the Wager-Whitin

property and in addition, whenever there is positive flow in

a production arc (J-T, J), there is positive flow on at least

one tranfer arc originating at J.

Our proof of this theorem will be similar to the proof of Theorem 1N; a flow

adjustment procedure will be described which yields the desired characteris-

tic without increasing total cost.

Initially we adjust the flow, if needed, to satisfy the WW property by

applying Algorithm 1. At this phase in our procedure, if there are some

distributor nodes that do not satisfy the nesting property, then to enforce

nesting we apply a flow adjustment operation, operation NSA. However, while

enforcing one desirable property , nesting, we should not disturb the other

desirable property, namely the WW property, which the flow already satisfies.

Therefore, in case the WW property is distrubed by operation NSA, it will be

restored by another operation, operation NSB.

We consider nodes from 2NT to 0 consecutively in decreasing order of in-

dex; when a node that does not satisfy nesting is encountered we apply first

NSA at that node to satisfy nesting, and then NSB, if needed to recover the

WW property. When we finish considering node 0, we will have a nested, WW

flow at hand.

Suppose we find in applying the above procedure that we need to apply

operation NSA at node J. Since J violates nesting, J must have incoming flow on its production arc and must not have outgoing flow on any of its transfer arcs. Also, since we are considering nodes in the descending order, all nodes greater than J must satisfy nesting, and the WW property. Since J violates nesting, and since the flow is feasible, J cannot belong to the first period nor to the last period of the planning horizon, but to some intermediate period. Therefore, there exists a future period t such that production can be shifted from node J to node Q, where Q is the distributor node for the stage under consideration in period t.

Such a shift of production-flow from J to Q necessitates flow changes in several arcs. The arcs along which changes are to be made are contained in subgraph $\sigma(J)$, the construction of which is described in Algorithm 2.1. The proposed shift in the production from J to Q is obtained by sending an adjusting flow around the subgraph $\sigma(J)$. The adjusting flow is so computed that the resulting flow satisfies the following properties: (i) the node J as well as all distributor nodes greater than J satisfy nesting and WW properties, (ii) all nodes satisfy the feasibility conditions, (iii) the cost of the resultant flow is not greater than that of the original flow.

However, in the resulting flow, some nodes with indices less than J may not satisfy the WW property after changing the flow. Therefore, we consider nodes from 1 through J-1 consecutively, applying operation WWA at those nodes where WW property is distrubed by operation NSA. Such consecutive application of operation WWA from 1 through J-1 is referred to as application of operation NSB at J. As the operation NSB does not distrub the flow beyond node J-1, after its application at J, the WW property holds throughout the network. The complete statement of the algorithm for obtaining a nested schedule is given in Appendix B.

To illustrate the procedure for obtaining a nested schedule, consider again the feasible flow shown in Figure 5. This flow satisfies the WW property, but the nested schedules property is violated at node 23. Operation NSA identifies the next period in which there is a positive flow on a transfer arc out of this stage (3), and then shifts production to that period. Shifting the production means that flows on the transfer arcs must also be shifted, resulting in (possibly additional) inventory carrying at the predecessor stages. Figure 6 shows the subgraph $\sigma(23)$ associated with the necessary flow changes.

After this flow change, the WW property will be violated at node 16. Applying operation WWA at node 16, we find that node 7 violates nesting. After applying operation NSA at node 7, the desired result is achieved. This flow is shown in Figure 7.

Proof of Theorem 2N

In a given feasible flow satisfying the WW property let J be a distributor node that does not satisfy the nesting property. Then application of operation NSA at J results in an adjusted-flow which (1) is feasible, (2) satisfies WW property at distributor nodes greater than or equal to J, (3) satisfies nesting property at J and at all distributor nodes greater than J, (4) has a cost less than or equal to the cost of flow prior to adjustment.

Of these four properties, the first three are easy to see; the last property is not so obvious and hence we will offer here a proof for it.

Note that in the operation NSA we shift production in the stage under consideration to a future period. As the production costs are non-increasing in time, the production costs are not going to increase as a result of this adjustment.
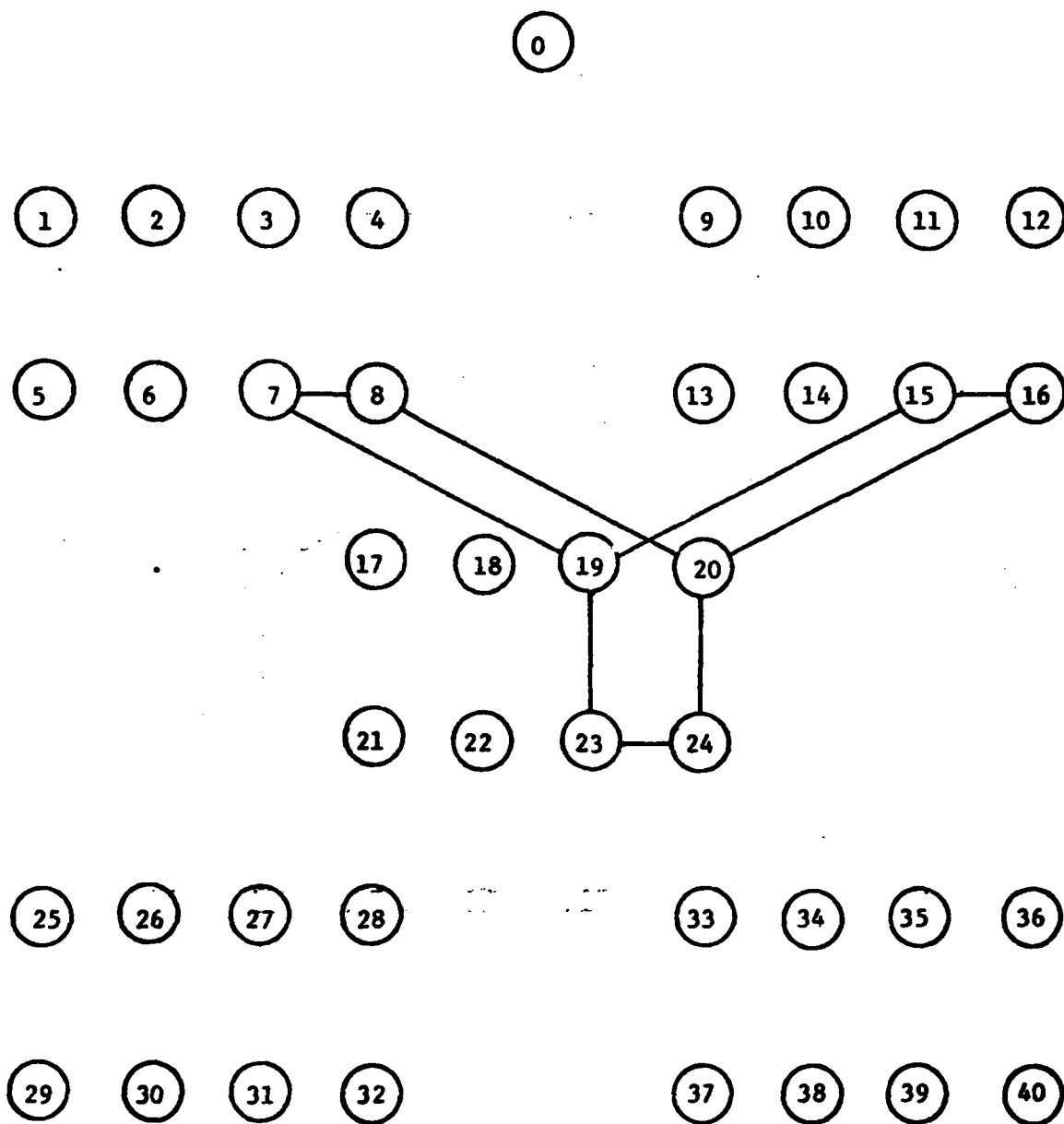
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Figure 6. The Subgraph σ(23)
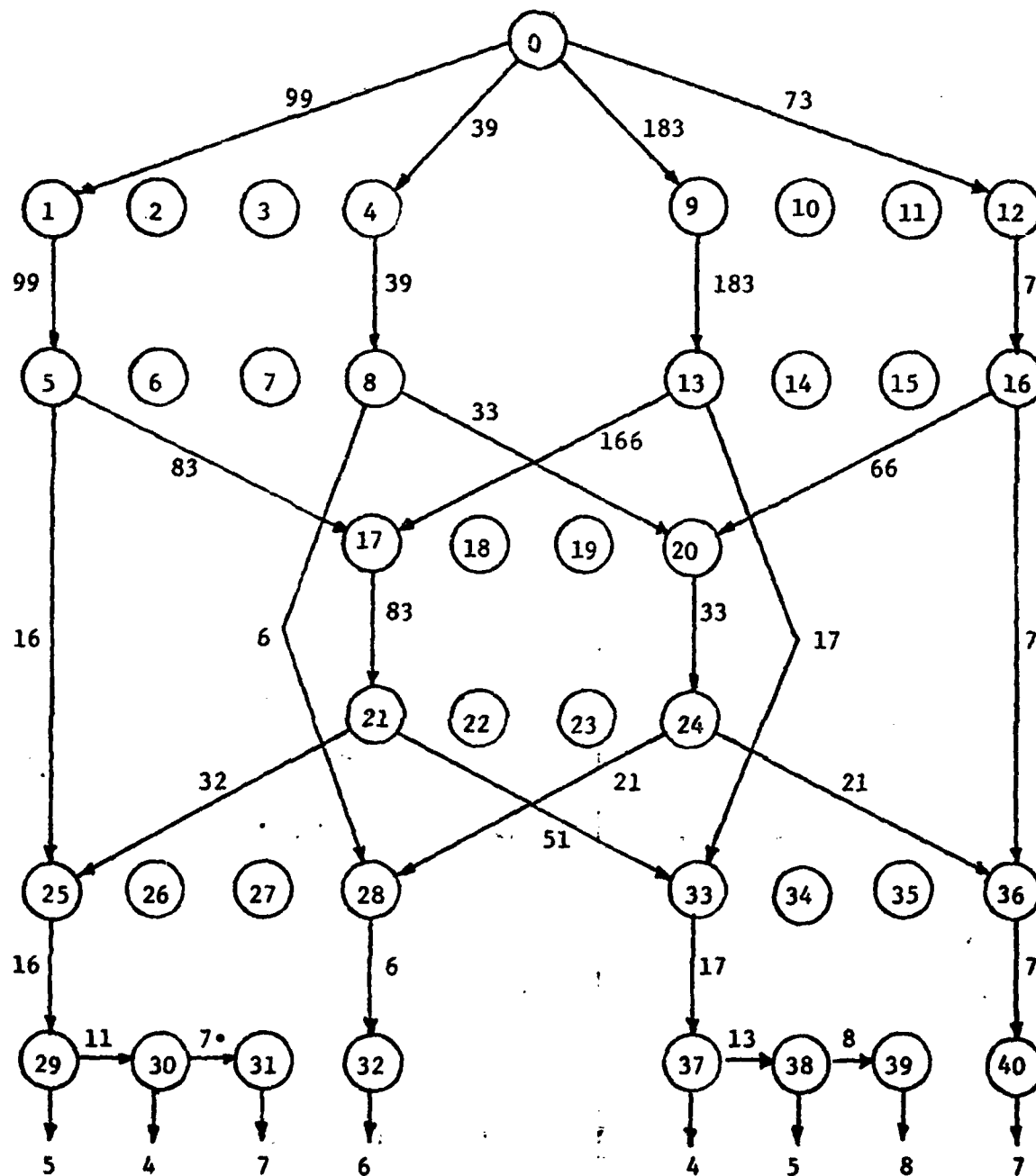
Figure 7. A Nested, WW Flow

The inventory costs remain to be considered, however. In the subgraph $\sigma(J)$, for every inventory arc of stage $i(J)$ there is one corresponding inventory arc in the same period for each immediate predecessor of $i(J)$. The flow in the inventory arcs of $i(J)$ decreases by U units and that in the inventory arcs of the predecessors increases by $K(.)U$ where $K(.)$ refers to the interstage multiplier between $i(J)$ and the predecessor stage under consideration.

Consider a particular inventory arc $(J+K, J+K+1)$. This arc represents inventory at stage $j = i(J)$ and period $t = t(J+K)$. Because the inventory costs are concave, nondecreasing, we have:

$$H_{gt}(K_{gj}X_{jt}) - H_{gt}(0) \geq H_{gt}(X_{gt} + K_{gj}X_{jt}) - H_{gt}(X_{gt}) \quad \forall \ g, \ t \quad \text{and}$$
$$j \in S(g)$$

Also, condition 4 of Theorem 2 gives

$$H_{jt}(X_{jt}) \geq \sum_{g \in P(j)} H_{gt}(K_{gj}X_{jt}).$$

Because $H_{jt}(0) = 0$ for all j and t, it follows that

$$H_{jt}(X_{jt}) - H_{jt}(0) \geq \sum_{g \in P(j)} \{H_{gt}(K_{gj}X_{jt}) - H_{gt}(0)\}$$

$$\geq \sum_{g \in P(j)} \{H_{gt}(X_{gt} + K_{gj}X_{jt}) - H_{gt}(X_{gt})\}$$

Thus, the inventory cost reduction in each period at stage $i(J)$ exceeds the inventory cost increase at the predecessor stages. Hence, operation NSA does not increase total costs. Q.E.D.

CONCLUSION

We have given constructive proofs of the Wagner-Whitin and Nested Schedule properties for optimal solutions to acyclic production systems with general integer multipliers and concave costs of production and inventory. These proofs are important for two reasons. First, since they are network-based, they provide greater intuition and insight into this complex problem. Second, because they are constructive, they provide a simple, efficient method for obtaining a local optimum, given any trial solution.

These results can be extended in an obvious way to allow backlogging at the final stages. The existance of initial inventories is difficult to formalize, but only affects these results in the early periods, until the initial inventories are consummed. The case of constant, identical lead times is trivial; proving these properties with nonidentical lead times will probably require stricter assumptions on the cost structures.

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APPENDIX A

Algorithm 1: (To establish the Wagner-Whitin Property)

<u>BEGIN</u>

   <u>FOR</u> J = 1 to 2NT <u>DO</u>

      <u>IF</u> J is a distributor <u>THEN</u>

         <u>IF</u> J violates WW property <u>THEN</u>

            · Apply operation-WWA at J (See Algorithm 1.1)

         <u>ENDIF</u>

      <u>ENDIF</u>

   <u>ENDFOR</u>

<u>END</u>

Algorithm 1.1 (Operation-WWA at a given node J)

<u>BEGIN</u>

      · Construct the subgraph $\Gamma_1(J)$ (Algorithm 1.1.1)

      · Compute the labels $L_1(m)$ for each node of $\Gamma_1(J)$
        and labels $l_1(m, n)$ for each arc of $\Gamma_1(J)$
        (Algorithm 1.1.2)

      · Construct the subgraph $\Gamma_2(J)$ (Algorithm 1.1.3)

      · Compute labels $L_2(m)$ for each node of $\Gamma_2(J)$ and
        labels $l_2(m, n)$ for each arc of $\Gamma_2(J)$
        (Algorithm 1.1.4).

   <u>IF</u> $F(1) \leq F(0)$ <u>THEN</u>

      · Adjust the flow in $\Gamma_1(J) \cup \Gamma_2(J)$ as below:

      $X(J-1, J) \leftarrow 0$

      $X(J-T, J) \leftarrow U+V$

      $X(m, n) \leftarrow X(m, n) + (l_1(m, n) - l_2(m, n))V$
            for all other $(m, n) \in \Gamma_1(J) \cup \Gamma_2(J)$

ELSE

• Adjust the flow in $\Gamma_1(J) \cup \Gamma_2(J)$ as below:

$X(J-1, \ J) \leftarrow U+V$

$X(J-T, \ J) \leftarrow 0$

$X(m, \ n) \leftarrow X(m, \ n) + (1_1(m, \ n) - 1_2(m, \ n))(-U)$
$\quad\quad$ for all other $(m, \ n) \ \varepsilon \ \Gamma_1(J) \cup \Gamma_2(J)$

ENDIF

END

Algorithm 1.1.1 (Construction of $\Gamma_1(J)$)

BEGIN

• Unmark all the arcs of the network

• Mark the arc $(J-T, \ J)$

• FOR $P = J-T$ to 0 DO

$\quad$ • IF at least one outgoing arc of P is marked AND
$\quad\quad$ there is at least one unmarked incoming arc
$\quad\quad$ with a positive flow THEN

$\quad$ • Mark all incoming arcs of P which have a
$\quad\quad$ positive flow

$\quad$ • ENDIF

• ENDFOR

• Include all marked arcs and their nodes in $\Gamma_1(J)$

END

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Algorithm 1.1.2 (Construction of labels in $\Gamma_1(J)$)

BEGIN

   $L_1(J) \leftarrow 1$

   FOR m = J to 1 DO

      IF m ε $\Gamma_1(J)$ THEN

         IF $L_1(m)$ is not yet computed THEN

            FOR each outgoing arc of m, (m, n), in $\Gamma_1(J)$ DO

               IF (m, n) is a transfer arc THEN

               $l_1(m, n) \leftarrow K(m, n) \cdot L_1(n)$

               ELSE

               $l_1(m, n) \leftarrow L_1(n)$

               ENDIF

            ENDFOR

$$L_1(m) \leftarrow \sum_{(m,n)\,\varepsilon\,\Gamma_1(J)} l_1(m, n)$$

         ENDIF

      ENDIF

   ENDFOR

END

Algorithm 1.1.3 (Construction of $\Gamma_2(J)$)

BEGIN

   • Unmark all arcs in the network

   • Mark the inventory arc (J-1, J)

   • FOR p = J-1 to 0 DO

      IF there is at least one outgoing arc of P which is marked

         AND there is at least one incoming arc of P with a

         positive flow THEN

      • Mark all incoming arcs of P which have a positive flow

      ENDIF

      ENDFOR

   Include all marked arcs in $\Gamma_2(J)$.

END

<u>Algorithm 1.1.4 (Construction of labels in $\Gamma_2(J)$)</u>

<u>BEGIN</u>

$L_2(J) \leftarrow 1$

<u>FOR</u> m = J to 0 <u>DO</u>

    <u>IF</u> $L_2(m)$ is not yet computed <u>THEN</u>

        <u>FOR</u> each outgoing arc of m, (m, n), in $\Gamma_2(J)$ <u>DO</u>

            <u>IF</u> (m, n) is a transfer arc <u>THEN</u>

                $l_2(m, n) \leftarrow K(m, n) \cdot L_2(n)$

            <u>ELSE</u>

                $l_2(m, n) \leftarrow L_2(n)$

            <u>ENDIF</u>

        <u>ENDFOR</u>

$$L_2(m) \leftarrow \sum_{(m,n)\in\Gamma_2(J)} l_2(m, n)$$

    <u>ENDIF</u>

  <u>ENDFOR</u>

<u>END</u>

APPENDIX B

Algorithm 2:   To establish the Nested Schedule Property

BEGIN

    FOR J = 2NT TO 1 DO

        IF J is a distributor node THEN

            IF J violates nesting property THEN

                Apply operation NSA at node J (See Algorithm 2.1)

                Apply operation NSB at node J (See Algorithm 2.2)

            ENDIF

        ENDIF

    ENDFOR

END

Algorithm 2.1 (To apply operation NSA at a distributor node J)

BEGIN

    • Let $U \leftarrow X(J-T, J)$

    • Find the distributor node Q in stage $i(J)$ such that

        (i) $t(Q) > t(J)$

        (ii)   At least one outgoing transfer arc of Q has

             a positive flow.

        (iii) The nodes between J and Q do not have flows

             on their outgoing transfer arcs.

    • Include inventory arcs (J, J+1), (J+1, J+2), ... ,

    (Q-1, Q) and their end nodes in the subgraph $\sigma(J)$.

    • Let the adjusted flow in arcs (J, J+1), (J+1, J+2),

    ... , (Q-1, Q) be zero, that is

      $X(J+n, J+n+1) = 0$ for $n = 0, 1, ... , Q-J-1$

    • Include arc (J-T, J) and node (J-T) in $\sigma(J)$

• Let the adjusted flow on (J-T, J) be zero, that is

    $X(J-T, J) = 0$

• Include arc (Q-T, Q) and node (Q-T) in $\sigma(J)$.

• Let the adjusted flow on (Q-T, Q) be

    $X(Q-T, Q) \leftarrow X(Q-T, Q) + U$

• <u>IF</u> $i(J) \notin F$ <u>THEN</u>

    <u>FOR</u> each distributor node m which is connected to

        (J-T) with a transfer arc (m, J-T) <u>DO</u>

        • Find the distributor node r in stage i(m) such

          that transfer arc (r, Q-T) exists.

        • Include transfer arc (m, J-T) in $\sigma(J)$.   •

        • Let X(m, J-T), the adjusted flow on (m, J-T),

          be zero.

        • Include the transfer arc (r, Q-T) in $\sigma(J)$

        • Let the adjusted flow on (r, Q-T) be

            $X(r, Q-T) \leftarrow X(r, Q-T) + U \cdot K(r, Q-T)$

        • Include inventory arcs (m, m+1), (m+1, m+2), ... ,

          (r-1, r) and the nodes m, m+1, ... , r in $\sigma(J)$

        • Let the adjusted flow along the inventory arcs

          between m and r be

            $X(m+n, m+n+1) \leftarrow X(m+n, m+n+1) + U \cdot K(m, J-T)$

                    for $n = 0, 1, ... , r-m-1$

    <u>ENDFOR</u>

<u>ELSE</u>

    Include transfer arcs (0, J-T) and (0, Q-T) and

    node 0 in $\sigma(J)$

    Let $X(0, J-T) \leftarrow 0$

    Let $X(0, Q-T) \leftarrow X(0, Q-T) + U$

<u>ENDIF</u>

<u>END</u>

<u>Algorithm 2.3</u> (Application of operation NSB at a given node J)

<u>BEGIN</u>

   <u>FOR</u> r = 0 to J-1 <u>DO</u>

      <u>IF</u> r is a distributor node <u>THEN</u>

         <u>IF</u> r does not satisfy the WW property <u>THEN</u>

            Apply operation WWA at r (see Algorithm 1.1)

         <u>ENDIF</u>

      <u>ENDIF</u>

   <u>ENDFOR</u>

<u>END</u>

Algorithm 2.3 (Application of operation NSB at a given node J)